Revival of oscillations from deaths in diffusively coupled nonlinear systems: Theory and experiment
Wei Zou, Michael Sebek, István Z. Kiss, and Jürgen Kurths

Citation: Chaos 27, 061101 (2017); doi: 10.1063/1.4984927
View online: http://dx.doi.org/10.1063/1.4984927
View Table of Contents: http://aip.scitation.org/toc/cha/27/6
Published by the American Institute of Physics

Articles you may be interested in
Complex behavior in chains of nonlinear oscillators
Chaos: An Interdisciplinary Journal of Nonlinear Science 27, 063104 (2017); 10.1063/1.4984800

Generalized synchronization between chimera states
Chaos: An Interdisciplinary Journal of Nonlinear Science 27, 053114 (2017); 10.1063/1.4983841

Behavioral synchronization induced by epidemic spread in complex networks
Chaos: An Interdisciplinary Journal of Nonlinear Science 27, 063101 (2017); 10.1063/1.4984217
The intrinsic oscillations of diffusively coupled oscillatory systems can be quenched via manifesting two structurally different oscillation quenching processes: amplitude death (AD) and oscillation death (OD) under diverse scenarios, such as a distribution of frequencies introduced by the diversity of individual subsystems or transmission delays due to a finite propagation speed of external signals. Even so, it is a well-known fact that robust oscillations are ubiquitous in many real-world systems, which are always reliably sustained as a prerequisite for their normal functional evolution. Thus, it is of great importance to unveil the underlying mechanisms of rhythmic activities against both AD and OD, which is deemed as a challenging issue of practical significance. Here, we report that a small time delay in the self-feedback component of the coupling can revoke AD and OD by destabilizing the corresponding stable steady states of diffusively coupled Stuart-Landau oscillators. We clearly demonstrate that such local self-feedback delay plays a key role in reviving oscillations from AD and OD, which is in sharp contrast to the propagation delay of external units with the tendency to suppress oscillations. We corroborate our results by studying the systems of coupled Stuart-Landau oscillators in different configurations of oscillation quenching: (i) AD in two instantaneously coupled oscillators with frequency mismatch, (ii) AD in two coupled identical oscillators with uniformly distributed propagation delays, (iii) AD to OD transition in two dynamically coupled oscillators, and (iv) AD in an arbitrary network of coupled oscillators with distributed propagation delays. Furthermore, the distinct dynamical roles of two temporal delays in the coupling have been well confirmed in an experiment of two diffusively coupled electrochemical reactions. Our study sheds a significantly new insight on the coupling delays in controlling dynamic activity of complex nonlinear systems, which will have a strong impact and invoke wide interests in the field of nonlinear dynamics as well as in various applications ranging from physics, biology, and engineering to social sciences.

I. INTRODUCTION

The studies of diffusively coupled oscillators have provided a rich source of ideas and insights to elucidate a plethora of intriguing and important self-organizing activities in a variety of areas ranging from physics, chemistry, biology and neuroscience to engineering. Oscillation quenching is an emerging behavior, which refers to the phenomenon of coupling-induced suppression of oscillatory dynamics. Two structurally distinct oscillation quenching phenomena have been distinguished, namely, amplitude death (AD) and oscillation death (OD). In the case of AD, oscillations are suppressed due to the stabilization of an otherwise unstable homogeneous steady state (HSS). Hitherto, conditions such as frequency mismatch, coupling with propagation delay, and dynamic and conjugate interactions, etc., have been identified to produce AD. In contrast, OD is manifested by stabilizing an inhomogeneous steady state (IHSS), where units populate different branches of the same stable IHSS.

In diverse realistic situations, stable and robust rhythmic activity is ubiquitously deemed to serve as a critical prerequisite for proper functioning of multitude natural and man-made systems, where the onset of AD and OD may strongly deteriorate their normal functional performances or even deteriorate their normal functional performances or even

4weizou83@gmail.com

1054-1500/2017/27(6)/061101/9/$30.00

27, 061101-1 Published by AIP Publishing.
result in a fatal breakdown, e.g., heart contraction, circadian rhythms, or visual information processing in mammalian brains.\textsuperscript{35–37} Quenching of oscillations, both AD and OD, in the above-mentioned real-life processes is harmful and undesirable. For this purpose, a few recent efforts have been devoted to the topic of reviving oscillations from deaths.\textsuperscript{38–44}

It is well known that time delays are a fundamental reality in natural systems. Physically, time delay involving the external sending units is commonly attributed to propagation of signals with a limited speed, which is widely recognized as propagation delay with the tendency to induce or facilitate the emergence of AD and OD.\textsuperscript{13–24} On the other hand, time delay associated with the internal local subsystem itself is identified as self-feedback delay, which is generally due to the fact that a system generally requires a finite time to sense the received information and then react to it.\textsuperscript{45} A particularly important application of internal self-feedback delays is vividly illuminated by the time-delayed autosynchronization in the area of control problems, where a noninvasive stabilization of a single uncoupled dynamical system by a time-delayed feedback loop has been investigated extensively.\textsuperscript{46,47} A significant amount of research has been done over the past few decades on the analyses of propagation delay in regulating dynamics of coupled nonlinear systems, where a large stride of progresses have been witnessed both theoretically and experimentally.\textsuperscript{51,52} In contrast, the potential impacts of an internal self-feedback delay of local subsystems in the coupling are comparatively less explored. There are only a few attempts devoted to the situation when the coupling involves an internal self-feedback delay. For example, the role of self-feedback delay in the coupling has been considered in the study of anticipated synchronization of unidirectionally coupled oscillators.\textsuperscript{48,49} where it has been shown well that the stable coupling interval of anticipated synchronization monotonically decreases as increasing the internal self-feedback delay. Recently, time-delayed feedback signals are further extended to stabilize an unstable torus in coupled heterogeneous systems.\textsuperscript{50}

The physical origins of external propagation delay and internal self-feedback delay are distinct. The internal self-feedback delay could also be induced by a finite reaction time, which may be given dynamically by the inertia of the local dynamical system without including an explicit delay term, such as the cavity delay time that is often negligible in the laser systems. However, the particular topological structure of a ring of coupled fiber lasers explicitly gives rise to this intrinsic self-feedback delay,\textsuperscript{53} where the external coupling delay is emphasized to be induced by a certain time of signal propagation, and the local self-feedback delay arises as an internal round-trip time for the light. So far, there still lacks a clear discrimination between them in regulating collective dynamics. Our aim in this work is to exploit their distinct roles in shaping collective rhythmic behaviors of coupled dynamical networks.

In this work, we exclusively identify distinct dynamical virtues of two types of temporal delays in the diffusive coupling and demonstrate that the local self-feedback delay of internal receiving units in the coupling plays a key role in reviving oscillations under various death scenarios that are previously reported to induce either AD or OD. In particular, even a tiny amount of local self-feedback delays is shown to efficiently revive oscillations from deaths in diffusively coupled networks. The propagation delay of external sending units can be utilized to suppress harmful oscillations, whereas the local self-feedback delay of internal receiving nodes prosperously serves as an underlying mechanism to sustain reliable rhythmic activities. To discriminate these different dynamical influences of two temporal delays in the diffusive coupling is of importance from theoretical aspects as well as practical applications, which further highlights that special attention should be paid to the particular temporal delays associated with external and internal units in the coupling.

II. TWO COUPLED STUART-LANDAU OSCILLATORS

A. AD with frequency mismatch

Let us start with a paradigmatic model of two coupled Stuart-Landau oscillators to elaborate our results

\[
\dot{Z}_j(t) = \left(1 + iw_j - |Z_j(t)|^2\right)Z_j(t) + K[Z_k(t - \tau) - Z_j(t - \delta)],
\]

(1)

\(j, k = 1, 2\) and \(j \neq k\), where \(Z_j = x_j + iy_j\) are complex variables, \(w_j\) is the intrinsic frequency of the \(j\)th uncoupled oscillator, and \(K\) quantifies the strength of coupling. In the coupling, \(\tau\) measures the propagation delay, which is generally due to a finite transmission speed of the signals from the sending oscillators \(Z_k\). In contrast, \(\delta\) accounts for the local self-feedback delay to receiving units \(Z_j\) in the coupling, which takes a finite time to sense and react to the information from its neighboring nodes. \(K = 0\), both uncoupled Stuart-Landau oscillators exhibit a stable limit cycle \(Z_j = e^{iw_jt}\). Previous studies of coupled systems (1) have already established that, for \(\delta = 0\), the unstable HSS \(Z_1 = Z_2 = 0\) can be stabilized within an appropriate interval of \(K > 0\) for a large frequency mismatch \(\Lambda = |w_1 - w_2| > 2\) without any propagation delay \(\tau = 0\) (Ref. 6) or for identical oscillators \(w_1 = w_2 = w\) in the presence of some propagation delay \(\tau > 0\);\textsuperscript{13} both scenarios lead to the emergence of AD with the collapse of the stable limit-cycle oscillations. Here, we will first reveal that the presence of a local self-feedback delay \(\delta\) of internal receiving nodes \(Z_j\) in the coupling alone is capable of reviving oscillations from AD under the above-mentioned two scenarios.

To show the role of \(\delta\) of \(Z_j\) in eliminating AD, one needs to examine the onset conditions of AD in the coupled system (1) by performing a standard linear stability analysis around \(Z_1 = Z_2 = 0\). Assuming all the perturbations to be varied as \(e^{\xi t} (\xi \in \mathbb{C})\), the corresponding characteristic equation determining the stability of HSS is given by

\[
(1 + iw_1 - Ke^{-i\delta} - \lambda)(1 + iw_2 - Ke^{-i\delta} - \lambda) - K^2 e^{2i\tau} = 0.
\]

(2)

AD occurs if and only if the largest real part of the eigenvalues \(\lambda\) is negative. It should be noted here that the presence of time delays creates the exponential terms into the
characteristic equation, which lead to having infinitely many roots with negative real parts but only a finite number of roots with positive real parts. In general, the explicit expressions for the roots of such transcendental characteristic equation cannot be solved analytically. To depict stability regions of AD in the parameter space, here, we resort to numerically computing the roots of the corresponding characteristic equations with the pseudo-spectral differentiation techniques described by Breda et al.\textsuperscript{34}

Without any delays $\tau = \delta = 0$, AD in the coupled system (1) has been well explored by Aronson et al.,\textsuperscript{6} who have analytically established that AD is stabilized within $1 < K < (1 + \Delta^2/4)/2$ if $\Delta = |w_1 - w_2| > 2$, i.e., the coupled system (1) experiences AD for a finite interval of coupling strength if the rotational frequencies of both oscillators are sufficiently disparate. From the results of Ref. 6, it should be emphasized that the difference of the two frequencies rather than their ratio contributes essentially to induce the emergence of AD. Next, we will probe the effect of $\delta$ on the onset of AD in the death scenario of frequency mismatch. Interestingly, for a given frequency mismatch $\Delta > 2$, we find that the stable coupling interval of AD monotonically decreases and eventually disappears as increasing the value of $\delta$ from zero; this is directly verified in Fig. 1(a) with $\Delta = 10$, where the frequencies are $w_1 = 10 - \Delta/2$ and $w_2 = 10 + \Delta/2$, and $\tau = 0$ is fixed. For a global picture, Fig. 1(b) further depicts the spread of stable AD regions in the parameter space of $(K, \Delta)$ for $\delta = 0$ (bounded by two black lines), 0.04 (red region), 0.05 (green region), and 0.06 (blue region). An unbounded stable AD region for $\delta = 0$ becomes an isolated island for $\delta > 0$, which gradually shrinks and no longer exists when $\delta$ increases beyond a certain threshold of $\delta_c$. Note that we have numerically confirmed that the stable AD region cannot reappear for larger values of $\delta > \delta_c$. Thus, the local self-feedback delay $\delta$ of receiving nodes in the coupling alone plays a crucial role in circumventing AD in coupled nonidentical oscillators.

B. AD with distributed propagation delays

When incorporating a propagation delay $\tau > 0$ in sending oscillators $Z_k$ in the coupling, Reddy et al. reported that AD occurs even for coupled identical oscillators $w_1 = w_2 = w$ in their pioneering work.\textsuperscript{13} If the propagation delays are further distributed within a certain interval, Atay found that AD is stabilized over a much larger parameter space compared with the case of a discrete propagation delay.\textsuperscript{15} To unveil the effect of local self-feedback delay $\delta$ of $Z_k$ in circumventing AD induced by propagation delay, we next study the more stringent and realistic case of two coupled Stuart-Landau oscillators with distributed propagation delays written as

$$
\dot{Z}_j(t) = (1 + iw_j - |Z_j(t)|^2)Z_j(t) + K \left[ \int_0^\infty f(t')Z_k(t - t')dt' - Z_j(t - \delta) \right].
$$

In the coupling, $f$ is an integral kernel describing a particular distribution of propagation delays, which is assumed to be positive-definite and normalized to unity. If $f$ is the Dirac’s delta function $f(t') = \delta(t' - \tau)$, the coupling recovers the form of a discrete propagation delay as the same in the coupled system (1). Note that the case with a discrete propagation delay has already been well investigated.\textsuperscript{60} Here, we discuss a uniformly distributed-delay kernel: $f(t') = 1/(2\pi)$ if $|t' - \tau| < \pi$ and zero elsewhere as considered in Ref. 15. $f$ degenerates to $\delta(t' - \tau)$ as $\pi \to 0$. The characteristic equation to determine the stability of HSS in the coupled system (3) with the above uniformly distributed-delay kernel can be derived as

$$
1 + iw - Ke^{-2\delta} + Ke^{-2\delta} \sinh(\lambda \pi)/\lambda \pi - \lambda = 0,
$$

whose roots are again numerically computed to obtain the stability regions of AD in the parameter space of $(\tau, K)$. Within the AD regime, all of the characteristic roots are in the open left-half complex plane. Atay has analysed this system with $w_1 = w_2 = w$ for $\delta = 0$ and reported that the AD regime is unbounded along the $\tau$-direction for $\pi > 0.008$ with $w = 30$.\textsuperscript{15} We have reproduced the stable AD region on the $(\tau, K)$ plane for $\pi = 0.02$ in Fig. 2(a), where the interesting tongue structure of the AD region is revealed. The tongue structures of the stability regions of AD in the parameter space of $(\tau, K)$ are commonly observed in different cases with not only distributed delays\textsuperscript{15,19} but also time-varying delays\textsuperscript{20,21} and multiple delays.\textsuperscript{22,23} The explicit mathematical description of such stability regions with tongue structures is still lacking, which is beyond the scope of this study. Here, our aim is to explore the effect of $\delta$ on the stability region of AD. Upon introducing $\delta$ of receiving nodes in the coupling, we astonishingly find that even with a tiny amount of $\delta$, the spread of the AD regime drastically shrinks as shown in Figs. 2(b) and 2(c) for $\delta = 0.005$ and 0.01, respectively. The AD region splits into three disconnected and bounded islands for $\delta = 0.012$ [Fig. 2(d)]. Decreasing $\tau$ further results in the AD region completely wiping off from the whole parameter space. Hence, the presence of the local self-feedback delay $\delta$ of receiving nodes in the coupling alone successfully circumvents AD in coupled oscillators even when the propagation delays of the sending units are distributed over a pronounced interval, corroborating the generic and robust nature of $\delta$ in reviving oscillations from AD.

---

**FIG. 1.** Revival of oscillations from AD in the coupled system (1) for $\tau = 0$. (a) The AD interval (black region) vs. the local self-feedback delay $\delta$ for the frequency mismatch $\Delta = 10$. (b) AD regions in the parameter space of $(\Delta, K)$ for $\delta = 0$ (bounded by the two black solid lines), 0.04 (red region), 0.05 (green region), and 0.06 (blue region), respectively. $w_1 = 10 - \Delta/2$ and $w_2 = 10 + \Delta/2$ are used.
C. AD to OD transition with dynamic coupling

The local self-feedback delay $\delta$ of receiving nodes in the coupling is capable of annihilating the onset of not only AD but also OD, and even the AD to OD transition. To illustrate this, we explore a system of two Stuart-Landau oscillators with dynamic coupling, which is expressed as

$$
\begin{aligned}
\dot{x}_j &= P_j x_j - wy_j + K(u_k(t) - x_j(t - \delta)), \\
\dot{y}_j &= wx_j + P_j y_j, \\
\dot{u}_k &= -u_k + x_k,
\end{aligned}
$$

(5)

where $P_j = 1 - x_j^2 - y_j^2, j, k = 1, 2$ and $j \neq k$. Here, the coupling term involves only the real parts of the Stuart-Landau oscillator, which is deemed to be necessary to induce a symmetry breaking in the systems of coupled Stuart-Landau oscillators.\(^5\) Besides the trivial HSS at the origin (0, 0, 0), the coupled system (5) has a non-trivial IHSS: $P(x_1^*, y_1^*, -x_2^*, -y_2^*), \text{ with } x_1^* = -wy_1^*/(w^2 + 2Ky_1^2)$ and $y_1^* = \sqrt{(K - w^2 + \sqrt{K^2 - w^2})/2K}$, which arises at $K = (w^2 + 1)/2$ via a pitchfork bifurcation. The presences of $\delta$ cannot disturb the structures of these steady states but may switch their stabilities.

By performing linear stability analyses of the above two types of fixed points, the characteristic equations to determine the stability of HSS (AD) and IHSS (OD) of the coupled system (5) are obtained as

$$
(1 - \lambda \pm \sqrt{1 - \lambda^2} \mp \sqrt{1 - \lambda^2} - \lambda \pm \sqrt{1 - \lambda^2} = 0,
$$

(6)

and

$$
(1 - 3x^2 - y^2 - \lambda \pm \sqrt{1 - x^2 - y^2} = 0
$$

\begin{equation}
\mp \sqrt{1 - x^2 - y^2} - \lambda
+ (w^2 - 4x^2y^2)(1 + \lambda) = 0,
\end{equation}

(7)

respectively. The occurrences of both AD and OD can be identified from the distributions of the corresponding characteristic roots on the complex plane, which can be easily computed out numerically with the pseudo-spectral differentiation techniques.\(^5\) For $\delta = 0$, the AD to OD transition is observed in two coupled identical oscillators of Eq. (5) with $w = 5$, which is shown in Fig. 3(a) by depicting the bifurcation diagram of the steady states. Three typical bifurcation diagrams with $\delta > 0$ are plotted in Figs. 3(b)–3(d), respectively. These bifurcation diagrams first depict the solutions of both HSS and IHSS as a function of $K$ only showing the real part $x_j$ via black dashed lines; then, the coupling intervals with stable HSS (IHSS) are portrayed by black (red) solid lines.
Intriguingly, with gradually increasing $\delta$, OD is destabilized first from large coupling strengths [Fig. 3(b) for $\delta = 0.01$], whose stable region totally vanishes at $\delta = 0.096$ [Fig. 3(c)], whereas the stable region of AD seems to be not affected. However, for further increasing $\delta$, the stable AD interval begins to decrease [Fig. 3(d) for $\delta = 0.15$]. Figure 3(e) plots the distributions of both stable coupling intervals of AD and OD as a function of $\delta$, which systematically characterizes the effect of $\delta$ in destabilizing the AD to OD transition. The continuity of the stable regions of AD and OD shown in Fig. 3(e) suggests that the bifurcation has a negligible effect on the stability of steady states. However, it can be clearly seen that increasing $\delta$ prefers to destabilize OD from large values of coupling strength first. Meanwhile, the lower bound of the stable coupling interval of OD keeps unchanged until to $\delta_{c,1} = 0.096$, at which OD is completely destabilized. Interestingly, for all values of $\delta \leq 0.096$, the stable AD interval remains the same as that of $\delta = 0$. Increasing $\delta$ beyond 0.096, the stable coupling interval of AD shrinks simultaneously from both its upper and lower bounds and eventually disappears at $\delta_{c,2} = 0.162$.

Figure 4 further plots the dependence of the two critical values of $\delta_{c,1}$ and $\delta_{c,2}$ on the intrinsic frequency $w$, which shows a clear monotonically decreasing relation. From the point of view of the stability analysis, AD and OD cannot be distinguished from each other, except for that the corresponding steady states have different characteristic eigenvalues. However, AD and OD are substantially different oscillation quenching phenomena in terms of the destabilization effect due to $\delta$. The local self-feedback delay $\delta$ of receiving nodes in the coupling revives oscillations from not only AD but also OD, and even the AD to OD transition by destabilizing first the stable IHSS and then the stable HSS step by step.

III. NETWORKED STUART-LANDAU OSCILLATORS

The local self-feedback delay $\delta$ in reviving oscillations from death carries over to an arbitrary network of coupled Stuart-Landau oscillators with arbitrarily distributed propagation delays. Let us consider a connected network of $N$ Stuart-Landau oscillators with distributed propagation delays governed by the equation

$$
\dot{Z}_j = \left(1 + iw_j - |Z_j|^2\right)Z_j + \frac{K}{\tilde{\gamma}_j} \sum_{k=1}^{N} g_{jk} \int_{\delta}^{\infty} f(\tau')Z_k(t - \tau')d\tau' - Z_j(t - \delta),
$$

(8)

where $j = 1, 2, \ldots, N$. The parameter set of $g_{jk}$ describes the coupling topology as the following: $g_{kk} = g_{kj} = 1$ if the $j$th and $k$th nodes are connected otherwise $g_{kk} = g_{kj} = 0$, $g_{ij} = 0$, and $d_j = \sum_{k=1}^{N} g_{jk}$ gives the degree of $j$th node. The function $f$ is a general distribution kernel of propagation delays.

From a linear stability analysis, the stability of the HSS (AD) in the coupled system (8) with $w_j = w$ is determined by the characteristic equation

$$
\lambda = 1 + iw - Ke^{-i\delta} + K\rho_j \int_{0}^{\infty} e^{-i\delta} f(\tau')d\tau',
$$

(9)

where $\rho_j$’s are the eigenvalues of $G = \left(\frac{\partial}{\partial \tau}\right)_{\tau=N}$ and can be ordered as $1.0 = \rho_1 \geq \rho_2 \geq \cdots \geq -\frac{1}{N-1} \geq \rho_N \geq -1.0$.55
The coupled network (8) experiences AD if and only if the largest real part of the roots of Eq. (9) for each $\rho_j$ is negative.

From the parametric approach in robust control theory, we infer that the stability region of AD is in fact decided only by two extreme eigenvalues $\rho_1 = 1$ and $\rho_N$. To intuitively explain the above statement, Fig. 5 depicts a typical type of the largest real part $\lambda_R$ of the characteristic equation (9) as a function of $\rho_q$, where the propagation delays $\tau'$ are uniformly distributed over $|\tau' - 1| < 0.07$, $\delta = 0.002$, $K = 5$, and $w = 10$ are fixed. For other parameters, quite similar structures are observed. Clearly, we notice that $\lambda_R$ monotonically increases as $|\rho_j|$ increases in both the half planes $\rho_j > 0$ and $\rho_j < 0$, with the turning point at $\rho_j = 0$ corresponding to the mode that is the most difficult to be destabilized. The above observation implies that the stability of HSS is finally determined only by the two network eigenvalues of $\rho_1 = 1$ and $\rho_N$, and the spread of the AD region is larger for larger values of $\rho_N$. Two types of network with $\rho_N = -1$ and $\rho_N = 0$ thus give the lower and upper bounds for stable regions of AD in arbitrary networks of coupled oscillators.

The upper limiting stability region of AD for the coupled networks is decided by $\rho_N = -1/(N - 1) \to 0$ ($N \to \infty$), which is exactly the case for networks with all-to-all topology. If there exists one characteristic equation (9) of $\rho_j$ having the positive real part, AD is impossible to occur. Therefore, to uncover the destabilizing effect of the local self-feedback delay $\delta$, it is enough to deal only with the stability of the characteristic equation (9) with $\rho_j = 0$, which gives a sufficient condition of $\delta$ for circumventing AD in the coupled network (8) with arbitrarily distributed propagation delays. The roots of the characteristic equation (9) with $\rho_j = 0$ can be implicitly resolved with the Lambert $W$ function

$$\lambda = \frac{1}{\delta} W(-\delta K e^{-(1+iw)\delta}) + 1 + iw. \quad (10)$$

Obviously, from the characteristic equation (9) with $\rho_j = 0$, a necessary condition for stabilizing HSS is $K > 1$ for $\delta = 0$. However, for each $K > 1$, the presence of the local self-feedback delay induces an instability if $\delta > \delta_c$ because a pair of complex conjugate eigenvalues cross the imaginary axis. The critical threshold $\delta_c$ can be analytically determined as

$$\delta_c(K) = \cos^{-1}(1/K) \quad (w + \sqrt{K^2 - 1}), \quad K > 1. \quad (11)$$

This theoretical prediction of $\delta_c(K)$ is plotted by the red lines in Figs. 6(a) and 6(b) for $w = 10$ and $w = 30$, respectively. The analytical upper boundary is well confirmed by the numerical results represented by the open circles obtained from Eq. (10). Note that both the analytical upper boundary $\delta_c(K)$ and the stability domain are obtained independent of the topology of network. The analytical expression of $\delta_c(K)$ predicts a maximum amount of $\delta$ in reviving oscillations from AD in an arbitrary network of coupled Stuart-Landau oscillators with generally distributed propagation delays. The analytical prediction by Eq. (11) is compared to the stability boundary of an arbitrary finite network; thus, the destabilization of AD already occurs for lower values of $\delta < \delta_c$ in certain networks, for instance in all bipartite networks, such as chain networks, star networks, grid networks, and tree networks where all have $\rho_N = -1$.

IV. EXPERIMENTAL RESULTS

In this section, experiments are presented with two delay-coupled electrochemical oscillators to provide evidence of reviving oscillations from AD, where the coupling with two different temporal delays is designed to follow the above theory. The schematic of the experimental setup is shown in Fig. 7(a). The oscillatory electrodissolution takes place in an electrochemical cell with two 1 mm diameter nickel wires (working electrode), an Hg/Hg$_2$SO$_4$/sat. K$_2$SO$_4$ reference electrode, and a Pt coated Ti counter electrode. The electrodes are immersed in 3 M sulfuric acid solution at 10°C. A multichannel potentiostat (Gill-IK64 ACM Instruments) sets the potential of the wires ($V_j$) (all potentials are given with respect to the reference electrode) and measures the currents ($i_j$). At constant potential and with $R_{\text{ind}} = 1\,k\Omega$ individual resistances attached to the nickel wires, the rate of dissolution, measured as current, exhibit oscillations that occur through a Hopf

![FIG. 5. Typical dependence of the largest real part $\lambda_R$ of the characteristic equation (9) on the network eigenvalue $\rho_j$.](image-url)

![FIG. 6. Revival of oscillations from AD in an arbitrary network of coupled Stuart-Landau oscillators (4) with arbitrarily distributed propagation delays. (a) and (b): The coupling interval containing the upper limiting region of AD as a function of $\delta$ for $w = 10$ and $30$, respectively.](image-url)
The oscillations arise because of the negative differential resistance of the metal dissolution due to the formation of surface oxides and inhibitory adsorption of bisulfate ions. A real-time LabView program collects the currents (a rate of 200 Hz) and converts these into electrode potentials \( E_j(t) \) by subtracting the ohmic drop from the circuit potential, i.e., \( E_j = V_j(t) - i_j(t) R_{\text{ind}} \). The electrode potentials are corrected for offset, \( E_j = E_j - \bar{o} \), where \( \bar{o} \) is a time averaged electrode potential. The feedback interface [Fig. 7(b)] perturbs a system parameter, the circuit potential, for each electrode independently to implement the coupling scheme

\[
V_j(t) = V_0 + K \left[ E_k(t - \tau) - E_j(t - \delta) \right],
\]

where \( j, k = 1, 2, j \neq k \), and \( K, \tau, \delta \) are the experimental coupling strength, propagation delay, and local self-feedback delay, respectively. Without any coupling \( (K = 0) \) at \( V_0 = 1070 \text{ mV} \) (about 5 mV above the Hopf bifurcation), the oscillations have slightly different natural frequencies \( \omega_1 = 0.372 \text{ Hz} \) and \( \omega_2 = 0.367 \text{ Hz} \) due to surface heterogeneities. As it is shown in Fig. 7(c), when the coupling is turned on \( (K = 1) \) with a propagation delay \( \tau = 0.28 \text{ s} \) but without a local self-feedback delay \( \delta = 0 \), the coupling with propagation delay successfully induces AD. However, when the local self-feedback delay is switched on to \( \delta = 0.14 \text{ s} \) (at \( t = 180 \text{ s} \)), the quenched oscillations are regained afterwards with a frequency of 0.432 Hz. We thus see that the frequency of the regained oscillations is about 17% different than the natural frequencies of uncoupled systems. This experimental observation of the revival of oscillations from AD induced by propagation delay qualitatively supports our theoretical prediction.

Furthermore, we have performed a series of experiments in which the local self-feedback delay \( \delta \) was slowly increased and we measured the mean peak-to-peak amplitude of the oscillations; the experimental observations are shown in Fig. 7(d). For small values of \( \delta < 0.11 \text{ s} \), the coupled system exhibits AD showing zero amplitude. However, once \( \delta \) passes a critical threshold, the oscillations are revived from AD, as demonstrated by nonzero amplitude with finite frequency. These experimental observations thus well confirm our theory that the local self-feedback delay in the coupling can revive oscillations from deaths.

V. DISCUSSIONS

In the death scenarios with \( \delta = 0 \), the diffusive coupling introduces a strong dissipation into the whole coupled systems resulting in the suppression of oscillations. Interestingly, the introduction of \( \delta \) in the coupling effectively weakens the effect of coupling delay \( \tau \) in facilitating the emergences of both AD and OD. An intuition mechanism of \( \delta \) in destabilizing AD and OD can be given as the following. Since the value of \( \delta \) in successfully revoking AD and OD is always very small, mathematically one can apply a linear approximation \( Z_j(t - \delta) \approx Z_j(t) - \delta \cdot dZ_j(t)/dt \). Then, the \( \delta \)-term enters into the coupled equations via the form of the product \( K \delta \cdot dZ_j(t)/dt \), which supplies an additional energy source to compensate the dissipation of diffusively coupled systems,
thus serving as an external driving force to excite oscillations by revoking AD and OD.

We would like to highlight that the oscillations revived from either AD or OD by δ occurs via a supercritical Hopf bifurcation in all the cases, which can be directly identified from the observation that a pair of conjugate characteristic eigenvalues cross the imaginary axis from left to right as gradually increasing δ beyond the critical value δc. Depending on the coupling strength K, the revived oscillations can be either in-phase or anti-phase. However, both the amplitudes and frequencies of the revived oscillations differ from those of the uncoupled systems. In this work, we have focused on revealing the destabilizing effect of δ on AD and OD. The detailed study that how the dynamics of revived oscillations are further affected by δ is not carried out, which constitutes an important and interesting direction in the future.

VI. CONCLUSION

In summary, we have exclusively identified the peculiar role of a local self-feedback delay δ of receiving units in the coupling. We have theoretically established that the presence of δ successfully revives oscillations from not only AD but also OD in the coupled Stuart-Landau oscillators. Theoretical predictions have been well supported by the experimental observations with two coupled electrochemical reactions, where it clearly corroborates the effect of δ in reviving oscillations from AD induced by propagation delay. This effect of a local self-feedback delay is in sharp contrast to that of propagation delay of sending nodes in the coupling, which is recognized to either induce or facilitate AD and OD. The local self-feedback delay of receiving oscillators is amenable to sustain reliable rhythmic activities under diverse death scenarios, manifested by even a tiny amount of δ efficiently annihilating the onset of not only AD but also OD, and even the AD to OD transition. An explicit distinction of the propagation delay τ of the senders and the local self-feedback delay δ of the receivers in the coupling could provide a more realistic and accurate representation of engineering and biological systems. A few prominent examples for such realistic systems may include the segmentation clock of vertebrates, the core pacemaker of the circadian clock, engineered systems of electronic circuits, etc. The framework of our study broadens our understanding of nontrivial roles of different coupling delays in regulating oscillatory dynamics of relevant systems in physics, technology, and life sciences.

ACKNOWLEDGMENTS

W.Z. acknowledges Prof. Lei-Han Tang and Prof. Changsung Zhou for the valuable discussions. This work was supported by the Hong Kong Scholars Program and the Research Grants Council of the HK SAR under Grant No. 12301514. I.Z.K. acknowledges the financial support from the National Natural Science Foundation CHE-1465013.